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The renormalization group, systems of units and the hierarchy problem

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Abstract
In the context of the renormalization group (RG) for gravity, I discuss the role of field rescalings and their relation to choices of units. I concentrate on a simple Higgs model coupled to gravity, where natural choices of units can be based on Newton's constant or on the Higgs mass. These quantities are not invariant under the RG, and the ratio between the units is scale-dependent. In the toy model, strong RG running occurs in the intermediate regime between the Higgs and the Planck scale, reproducing the results of the Randall–Sundrum I model. Possible connections with the problem of the mass hierarchy are pointed out.

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1. Introduction

In a Wilsonian approach to the renormalization group (RG), physics at some energy scale k is described by an effective action Γ_k which can be regarded as the result of having functionally integrated all field fluctuations with momenta larger than k [1]. Thus k can be interpreted as an infrared cutoff and the RG describes the dependence of the action on this cutoff. An infinitesimal RG transformation is basically an integration over fluctuations of the fields with momenta ranging from k to $k - \mathrm{d}k$. A sequence of infinitesimal RG transformations defines a flow in the space of all possible actions. I will call this the 'basic' RG flow.

In quantum field theory (QFT), one can use the freedom of redefining the fundamental fields to eliminate some couplings from the action. The most familiar example, and the one that we will be dealing with in this paper, is normalizing the coefficient of the kinetic term by means of a global rescaling of the fields. If one starts from such a normalized action and performs a RG transformation, the resulting action will generally no longer be normalized, but it can be brought back to normalized form by means of a field redefinition. It is therefore customary to include in the definition of a RG transformation also this field redefinition. It is also convenient (and for numerical simulations even necessary) to work with dimensionless

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variables; this can be achieved by multiplying every coupling by a suitable power of k, i.e. using k as a unit of mass. (Similarly, in QFT on a lattice one usually takes the lattice spacing as a unit of length.) The RG transformation changes the value of the cutoff, and to compensate this it is customary to include in the definition of a RG transformation also a rescaling of lengths and momenta.

Thus, the textbook definition of a 'complete' RG transformation in QFT or statistical mechanics involves three steps: (i) functional integration over field fluctuations, (ii) rescaling of the field, (iii) rescaling of coordinates and momenta. This defines a flow in the space of normalized actions. In this paper, I discuss the use of field rescalings in the presence of gravity. Because of the special geometrical status of the metric field, its rescalings can be interpreted as changes in the unit of length. Therefore, when the metric is dynamical, step (iii) is just a special case of step (ii), applied to the metric. Furthermore, normalizing the metric is equivalent to a choice of units. Different choices of units lead to different RG flows and, as a consequence of the rescalings, the normalized metric becomes scale-dependent. This discussion will be used to clarify some points related to the RG flow in the theory of gravity. In particular, it will be shown that earlier results on the RG flow in Einstein's theory will not be affected by consideration of these rescalings.

In the second part of the paper, I will use these insights to show how certain five-dimensional models can be seen as geometrical reformulations of the four-dimensional RG flow. In order to make the discussion more concrete, we will analyse in some detail a toy model consisting of gravity coupled to a scalar field. Assuming that this model reflects some features of the real world, the scalar field will be referred to as the 'Higgs field' and its physical mass will be assumed to be in the TeV range. In this model, various choices of units are possible[1]: cutoff units, Planck units based on Newton's constant and Higgs units based on the mass of the scalar field, each leading to a different form of the RG flow. The conversion ratio between Higgs and Planck units is

$$\alpha = \frac{m_H}{m_P} = m_H\sqrt{G} \approx 10^{-16}. \qquad (1)$$

Insofar as these units are built with fundamental physical variables, the appearance of such a small ratio is puzzling[2]. This was noted already long ago by Dirac, who considered various very large numbers occurring in Nature [3]. One such number was the ratio of the gravitational to electromagnetic force between an electron and a proton. Dirac's response was to assume that a very large number can only arise as a result of another number being very large. Another very large number which turns out to be of comparable magnitude is the age of the universe, τ, expressed in atomic units. Dirac postulated that this rough coincidence is due to some as yet undiscovered law, and that to maintain it in the course of time Newton's constant, when measured in atomic units, would decrease as τ^{-1}. Experimental bounds are now strong enough to rule out this behaviour (for a review of observations see e.g. [4]).

The mass of the electron and the mass of the proton have quite different origins, and for our discussion it will be convenient to replace Dirac's original electron and proton by any pair of charged pointlike particles (quarks or leptons). The mass of all these particles is believed to originate from their Yukawa interaction with the Higgs VEV; likewise, the Higgs mass m_H originates from the interaction of the scalar quanta with the VEV. Thus the ratio (1) is a good representative of the problem posed by the smallness of the mass of any known pointlike particle, in Planck units. It is also closely related to Dirac's problem, because the ratio of

[1] As is customary in QFT, we take $\hbar$ as a unit of action and c as a unit of velocity. With these choices, everything has the dimension of a power of length (or mass). A system of units is then defined by choosing a standard of length (or mass).

[2] See [2] for recent discussions about fundamental constants and fundamental units.

gravitational to electric forces for pairs of charged pointlike particles, up to dimensionless couplings, is essentially the square of (1).

In particle physics the smallness of (1) is known as the hierarchy problem. In perturbative QFT, m_H^2 can be thought of as the sum of a classical ('bare') value and a quantum correction. Since the dominant graph for the radiative correction of m_H^2 is quadratically divergent, the quantum correction is proportional to the square of the UV cutoff, which is presumably of the order of Planck's mass. In order for the observed value to be so much smaller than the Planck mass, the bare value should cancel the quantum correction with an extraordinarily high precision. While not necessarily 'wrong', this fine tuning is very artificial and should probably be taken as a hint that the perturbative approach does not describe properly what is happening.

Some years ago Randall and Sundrum (RS) proposed an elegant approach to the hierarchy problem, based on a classical five-dimensional model [5]. They start from the five-dimensional dynamics of gravity coupled to two parallel three-branes. It is assumed that matter fields are somehow confined to one of the branes, which is to be identified with our four-dimensional world, while gravity propagates in the five-dimensional bulk. This configuration, with a five-dimensional anti-de Sitter (AdS) metric between the branes, can be made to solve the equations of motion provided the cosmological constants in the bulk and in the branes are properly matched. It then emerges that any mass parameter in the 'physical' brane, measured relative to the four-dimensional metric which solves Einstein's equations, is suppressed by an exponential factor with respect to the fundamental parameters appearing in the action.

Returning now to the RG flow, we will see that the equations for the running parameters of the toy model in certain approximations are identical to those found by RS in [5]. With hindsight, this is not surprising: based on ideas that originate from the AdS/CFT correspondence [6], the RS model has been given a 'holographic' interpretation, whereby the fifth coordinate in the anti-de Sitter (AdS) space can be regarded as the logarithm of a RG scale [7]. The five-dimensional slice of AdS space that is used in the RS model reproduces precisely the leading scaling behaviour of the flat metric when the RG equations are written in cutoff or Higgs units. There is however a fundamental difference in interpretation. The RG equations used in this paper are not derived from the classical higher-dimensional Einstein equations: instead, they are calculated from first principles in the four-dimensional QFT and the five-dimensional AdS metric is merely a suggestive auxiliary construction. In fact, the dynamics of gravity plays a relatively little role in this discussion: the contribution of gravitons to the beta functions is suppressed in the regime that is of interest for the hierarchy problem, and the main role of gravity is to set the initial conditions for the RG evolution. Nevertheless, it will be clear that the natural framework for this discussion is in the presence of dynamical gravity.

This paper is organized as follows. Section 2 discusses the invariances of the action under scalings of the fields, the notion of 'redundant' or 'inessential' couplings and some peculiarities of the RG applied to gravity. In section 3, these notions are illustrated in a toy model of gravity coupled to a scalar field; various choices of units are discussed, each leading to a different form of the RG equations. Section 4 contains the actual behaviour of the ratio α as a function of k in the toy model. In section 5 the RS model is reconstructed starting from the four-dimensional RG flow; some general consequences of the running of the units are illustrated. Finally in section 6, I discuss the use of dimensionful quantities in quantum gravity, possible connections of the RG flow to the hierarchy problem and I present some final comments and conclusions.

2. Field rescalings and redundant couplings

Let us assume that physics at a certain energy scale k is described, to the desired degree of accuracy, by an effective action Γ_k, a functional depending on certain fields ϕ_A and coupling constants g_i[3]. The action describing the physics at some lower energy scale $k - \mathrm{d}k$, is given by a functional integral over all fluctuations of the fields with momenta in the range $k - \mathrm{d}k < |q| < k$, using Γ_k as the classical action. In general, even if Γ_k contained only a finite number of terms, this functional integral will produce an effective action $\Gamma_{k-\mathrm{d}k}$ with infinitely many effective couplings. Therefore, Γ_k should be thought of from the outset as 'the most general action' for the fields. It will have the form

$$\Gamma_k(g_i, \phi_A) = \sum_i g_i(k)\mathcal{O}_i(\phi_A), \tag{2}$$

where $\mathcal{O}_i$ are all operators constructed with the fields and their derivatives, which are compatible with the symmetries of the theory and belong to a given class of functionals (e.g. one may or may not require locality). The dependence of the effective action on k is given by

$$\partial_t\Gamma_k(g_i, \phi_A) = \sum_i \beta_i(g_j, k)\mathcal{O}_i(\phi_A), \tag{3}$$

where $t = \log(k/k_0)$ (for some arbitrary initial value k_0) and $\beta_i(g_j, k) = \partial_t g_i$ are the 'basic' beta functions, describing the change of the effective action when one integrates out fluctuations of the fields with momenta between k and $k - \mathrm{d}k$. Note that since only an infinitesimal range of momenta is involved, the beta functions defined in this way are finite, independently of the UV behaviour of the theory.

Once the beta functions in (3) are known, one can start from any arbitrary initial point and follow the RG trajectory in either direction. It happens very often that the flow cannot be integrated towards the UV beyond a certain limiting scale Λ. In this case the theory can only be used for $k < \Lambda$; it is called an 'effective theory' or a 'cutoff theory' and Λ is the physical cutoff (as opposed to a mathematical cutoff, which is used to regulate divergences, or to k, which is an artificial device used to compute the scale dependence of the action). It may happen, however, that the limit $t \to \infty$ can be taken; in this case the theory is said to be 'fundamental'. A fundamental QFT is a self-consistent description of a certain set of physical phenomena which is valid up to arbitrarily high-energy scales and does not need to refer to anything else outside it.

We can think of the space of all theories as an infinite-dimensional manifold $\mathcal{Q}$, whose coordinates are the coupling constants g_i appearing in the action. Similarly, let us denote $\mathcal{F}$ the infinite-dimensional space of all field configurations, whose coordinates are the fields $\phi_A(x)$. We can think of $\Gamma_k(\phi_A, g_i)$, for fixed k, as a functional on $\mathcal{F} \times \mathcal{Q}$. This description generally contains more parameters than are strictly necessary to describe the physics. The fields ϕ_A are integration variables, and a redefinition of the fields does not change the physical content of the theory. This can be seen as an arbitrariness in the choice of coordinates in $\mathcal{F}$. There is a similar arbitrariness in the choice of coordinates on $\mathcal{Q}$, due to the freedom of redefining the couplings g_i. We will restrict the group $\mathcal{G}$ of allowed field redefinitions in such a way that the transformed action belongs to the same class of functionals as the original action. Since Γ_k is the most general functional in this class, given any field redefinition $\phi' = \phi'(\phi)$, one can find a new set of couplings g'_i such that, for fixed k,

$$\Gamma_k(\phi'(\phi), g) = \Gamma_k(\phi, g'). \tag{4}$$

[3] The word 'effective' is used here in the same sense as in 'effective field theory'.

This defines an action of $\mathcal{G}$ on $\mathcal{Q}$. We are free to choose a coordinate system on $\mathcal{Q}$ which is adapted to these transformations, in the sense that a subset $\bar{g}_{\bar{\imath}}$ of couplings are invariant under the action of the group while a subset $\hat{g}_{\hat{\imath}}$ transform nontrivially. The couplings $\hat{g}_{\hat{\imath}}$ are called redundant or inessential, while the couplings $\bar{g}_{\bar{\imath}}$ are called essential. (It is important to stress that this distinction is parametrization dependent; in the appendix, I illustrate this point with a few specific examples.) A theory will generally contain infinitely many essential and infinitely many inessential parameters, and all physical observables can be expressed as functions of the essential couplings only.

One can exploit reparametrization invariance to eliminate the redundant couplings and follow only the flow of the essential ones. In general in an effective QFT there will still be infinitely many couplings to follow, but in approximation schemes where only a finite number of couplings are retained this may be a useful way of reducing the number of variables. Field redefinitions also play an important role in discussions of composite operators, see e.g. [8–10]. Mathematically, the 'basic' RG flow defined on the space $\mathcal{Q}$ projects onto a flow in the quotient space $\bar{\mathcal{Q}} = \mathcal{Q}/\mathcal{G}$. In practice, the flow on $\bar{\mathcal{Q}}$ can be described explicitly by giving the complete flow of the essential couplings. It is characterized by beta functions which differ from the 'basic' beta functions by suitable infinitesimal field redefinitions, as discussed in the introduction. Such redefinitions are proportional to the beta functions of the redundant couplings and are usually written in terms of the anomalous dimensions, defined as

$$\eta_{\hat{\imath}} = \frac{\beta_{\hat{\imath}}}{g_{\hat{\imath}}} = \partial_t \log \hat{g}_{\hat{\imath}}. \tag{5}$$

Of all field redefinitions, the ones that will interest us in this paper are the constant rescalings. It is generally the case that for each field ϕ_A (or more precisely for each multiplet of fields) there is a scaling invariance of the action, provided also the couplings are suitably transformed. By means of this scaling, for each field multiplet one can eliminate one coupling from the action. The usual choice is to eliminate the wavefunction renormalization constant Z_A, which is the coefficient of the kinetic term of the field multiplet.

In quantum gravity, the RG flow has some special features that are not present in other QFTs. This is due to the special meaning of the metric, which is the field used to measure lengths. As in any theory, the action is invariant under rescalings of the metric, provided all fields and couplings are also rescaled suitably:

$$S(g_{\mu\nu}, \phi_A; g_i) = S(b^{-2} g_{\mu\nu}, b^{d_A}\phi_A; b^{d_i} g_i). \tag{6}$$

In this equation, d is the canonical dimension of each field or coupling, in units of mass. Thus, dimensional analysis is a statement of invariance of the action under rescalings of the metric. Note that this definition of canonical dimension holds whether or not the metric is treated as a dynamical variable, but the freedom to redefine the metric really only arises in quantum gravity. This b-invariance can be seen as the mathematical manifestation of the fact that dimensionful quantities cannot, strictly speaking, be measured: one can only measure dimensionless ratios of dimensionful quantities. By a measurement of a dimensionful quantity one really means the measurement of the ratio of the dimensionful quantity and a suitably defined unit. To choose a system of units is to break the scale invariance of dimensional analysis.

Scale invariance is usually assumed to be broken in QFT, due to the need to introduce the mass scale k (or in other approaches, a UV cutoff or a renormalization point). However, if k is rescaled by a factor b (as its mass dimension requires), scale invariance is maintained also in the quantum theory:

$$\Gamma_k(g_{\mu\nu}, \phi_A; g_i) = \Gamma_{bk}(b^{-2} g_{\mu\nu}, b^{d_A}\phi_A; b^{d_i} g_i). \tag{7}$$

If one did not transform the cutoff k, then there could be an anomaly. The two situations correspond therefore to two different realizations of the same abstract group of scale transformations. Dimensional analysis is based on the realization where scale invariance is preserved.

As mentioned in the introduction, the standard definition of the RG in QFT involves (ii) a constant rescaling of the field and (iii) a constant rescaling of coordinates and momenta. A rescaling of space can be treated equivalently as a rescaling of the metric; in the former case, one is implicitly assuming that the coordinates have dimension of length and $g_{\mu\nu}$ is dimensionless, while in the latter the coordinates are dimensionless, as is more natural in general relativity, and the metric has dimension of length squared. There follows that in the presence of dynamical gravity step (iii) can be seen naturally as a special case of step (ii) applied to the metric field. Furthermore, in the case of pure gravity, the rescalings (ii) and (iii) are not independent operations, and as a result one can use them to eliminate only one coupling, instead of two, as is the case in ordinary QFTs [11].

A related point is that in the case of gravity, the group $\mathcal{G}$ of field redefinitions affects also the parameter k. It is therefore convenient to treat k as one of the couplings of the theory. Formally, we can think that the space $\mathcal{Q}$ contains a factor R^+ parametrized by k. When k is treated as a coupling, we can ask whether it is essential. As discussed above, this will depend on the choice of parametrization of $\mathcal{Q}$. Due to the scaling invariance (7) there will always be at least one redundant (dimensionful) coupling, which can be eliminated by a choice of units. The standard choice in QFT is to take k as a unit of mass. This means choosing coordinates $\{k, \tilde{g}_i\}$, where $\tilde{g}_i = g_i k^{-d_i}$ are the couplings measured in units of the cutoff. In this parametrization, k is a redundant coupling and the $\tilde{g}_i$ are invariant under the rescalings (7). With this choice k never appears explicitly in any physical observable, nor in the beta functions, so that the RG flow is given by an autonomous system of differential equations.

Alternatively, we can also pick any coupling h of dimension $d \neq 0$ and take $h^{1/d}$ as a unit of mass. Then we choose coordinates $\{h, \hat{k}, \hat{g}_i\}$, where $\hat{k} = kh^{-(1/d)}$ and $\hat{g}_i = g_i h^{-(d_i/d)}$ are the cutoff and the other couplings measured in units of h. In this parametrization h is redundant and the other dimensionless coordinates are invariant under the rescalings (7). Which one of them is essential then depends upon the way in which the remaining field reparametrizations act; however, in these units $\hat{k}$ is essential and physical observables may depend on it.

3. RG and systems of units in a toy model

In this section, I will illustrate various forms of the RG in a toy model consisting of gravity coupled to a real scalar 'Higgs' field ϕ. Besides serving as an illustration of the general concepts described in the previous section, such a model also contains the main ingredients that enter into the hierarchy problem. While very incomplete in some respects, it may still be sufficiently realistic to capture some of the physical effects that play a role in the generation of the hierarchy. It is also useful for the comparison with [5].

We will deal with actions of the general form (metric signature is $-+++$)

$$S(g_{\mu\nu}, \phi; \Lambda, Z_g, Z_\phi, m_H, \lambda, \xi) = \int \mathrm{d}^4x\sqrt{g}\left[\Lambda + Z_g R[g] - \frac{Z_\phi}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\lambda(\phi^2 - \upsilon^2)^2 + \xi\phi^2 R[g]\right]. \quad (8)$$

Here $Z_g = \frac{1}{16\pi G}$ is, in the linearized theory, the wavefunction renormalization of the graviton. The potential V is assumed to have the familiar quartic form with nonvanishing minimum. It can be parametrized by the value of the VEV, υ, by the Higgs mass $m_H^2 = \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\big|_{\phi=\upsilon}$ and the quartic coupling $\lambda = \frac{1}{12}\frac{d^4V}{d\phi^4}\Big|_{\phi=\upsilon}$. These parameters are related by $m_H^2 = 4\lambda\upsilon^2$, so we can choose m_H and λ as independent parameters and treat υ as a function of the other two.

Our arguments in the following sections will be based on the assumption that in the relevant range of energies the effective action Γ_k can be well approximated by a functional of the form (8), where the couplings Λ, Z_g, Z_ϕ, m_H^2, λ and ξ are now k-dependent (we omit to write this dependence explicitly for notational simplicity). The effective action has the following two scaling properties:

$$\Gamma_k\big(g_{\mu\nu}, \phi; \Lambda, Z_g, Z_\phi, m_H^2, \lambda, \xi, \ldots\big)$$
$$= \Gamma_{bk}\big(b^{-2}g_{\mu\nu}, bc^{-1}\phi; b^4\Lambda, b^2 Z_g, c^2 Z_\phi, b^2c^2 m_H^2, c^4\lambda, c^2\xi, \ldots\big) \tag{9}$$

where b and c are positive real numbers and the dots stand for higher order terms that we are neglecting. The transformations with parameter c express the behaviour of the theory under rescalings of the scalar field. The transformations with parameter b are the expression of dimensional analysis and are a special case of (7).

Because the action (8) has two scaling invariances, one can use them eliminate two combinations of the couplings from the action. The parameter c can be used to fix any coupling that multiplies an operator containing the field ϕ. The standard choice is to fix $Z_\phi = 1$. We will always stick to this choice in the following (for an alternative see the appendix). The parameter b can be used to fix any one of the dimensionful couplings. By definition, that coupling then defines a unit of length and mass. Let us now consider various choices of units.

3.1. Cutoff units

In RG theory the standard choice is to take the cutoff k as a unit of mass. By means of a scaling (9) with parameters $b = 1/k, c = 1/\sqrt{Z_\phi}$ one can define the normalized action in cutoff units

$$\tilde\Gamma\big(\tilde g_{\mu\nu}, \tilde\phi; \tilde\Lambda, \tilde Z_g, \tilde m_H^2, \tilde\lambda, \tilde\xi, \ldots\big) = \Gamma_1\big(\tilde g_{\mu\nu}, \tilde\phi; \tilde\Lambda, \tilde Z_g, 1, \tilde m_H^2, \tilde\lambda, \tilde\xi, \ldots\big)$$
$$= \Gamma_k\big(g_{\mu\nu}, \phi; \Lambda, Z_g, Z_\phi, m_H^2, \lambda, \xi, \ldots\big), \tag{10}$$

where

$$\tilde g_{\mu\nu} = k^2 g_{\mu\nu}, \qquad \tilde\phi = \frac{\sqrt{Z_\phi}}{k}\phi, \qquad \tilde\Lambda = \frac{1}{k^4}\Lambda, \qquad \tilde Z_g = \frac{1}{k^2}Z_g,$$
$$\tilde m_H^2 = \frac{1}{Z_\phi k^2}m_H^2, \qquad \tilde\lambda = \frac{1}{Z_\phi^2}\lambda, \qquad \tilde\xi = \frac{1}{Z_\phi}\xi, \ldots. \tag{11}$$

In this parametrization k and Z_ϕ are inessential and do not appear among the arguments of $\tilde\Gamma$; the anomalous dimension of the scalar field is defined according to (5) as $\eta_\phi \equiv \eta_{Z_\phi} = \partial_t \log Z_\phi$. As noted in [11], there is not enough freedom to eliminate k and Z_g at the same time, so with this choice of units the action $\tilde\Gamma$ (and also the beta functions, and physically measurable quantities) depends explicitly on $\tilde Z_g = \frac{1}{16\pi\tilde G}$, where $\tilde G = k^2 G$ can be viewed as Planck's area measured in cutoff units.

An infinitesimal RG transformation of $\tilde\Gamma$ then consists of the followings.

(1) Functional integration over a shell of momenta, from k to $k + \delta k$, with $\delta k < 0$. This produces a variation of the couplings $\delta g_i = \beta_i\delta t$. Since k has changed by a factor $1 + \delta t$ and Z_ϕ has changed by a factor $1 + \eta_\phi\delta t$, this has to be followed by
(2) a rescaling as in (9) with parameter $b = 1 - \delta t$ and
(3) a rescaling as in (9) with parameter $c = 1 - \frac{1}{2}\eta_\phi\delta t$, restoring the value $Z_\phi = 1$.

3.2. Planck units

Another possibility is to choose $\sqrt{16\pi Z_g} = G^{-1/2} = m_P$ as the unit of mass. Being based on the fundamental constants that enter in special relativity, in the quantum theory and in the theory of gravity, these units are often considered 'the most fundamental ones' [2]. A transformation with parameters $b = 1/\sqrt{16\pi Z_g} = \sqrt{G}$, $c = 1/\sqrt{Z_\phi}$ can be used to define the action in Planck units:

$$\begin{aligned}\Gamma'_{k'}\big(g'_{\mu\nu}, \phi'; \Lambda', m'^2_H, \lambda', \xi', \dots\big) &= \Gamma_{k'}\left(g'_{\mu\nu}, \phi'; \Lambda', \frac{1}{16\pi}, 1, m'^2_H, \lambda', \xi', \dots\right)\\ &= \Gamma_k\big(g_{\mu\nu}, \phi, \Lambda, Z_g, Z_\phi, m^2_H, \lambda, \xi, \dots\big),\end{aligned} \tag{12}$$

with

$$\begin{array}{llll} k' = \frac{k}{\sqrt{16\pi Z_g}}, & g'_{\mu\nu} = 16\pi Z_g g_{\mu\nu}, & \phi' = \sqrt{\frac{Z_\phi}{16\pi Z_g}}\phi, & \\ \Lambda' = \frac{\Lambda}{(16\pi Z_g)^2}, & m'^2_H = \frac{m^2_H}{16\pi Z_g Z_\phi}, & \lambda' = \frac{\lambda}{Z^2_\phi}, & \xi' = \frac{\xi}{Z_\phi}. \end{array} \tag{13}$$

As noted in [11], there is not enough freedom to eliminate k and Z_g at the same time, so with this choice of units the action Γ' depends explicitly on k', which can be viewed as the cutoff measured in Planck units. This is rather unusual, because the flow equations become non-autonomous. Let us denote $\eta_g \equiv \eta_{Z_g} = \partial_t \log Z_g$ the anomalous dimension of the graviton.

An infinitesimal RG transformation of Γ' consists of the following:

(1) functional integration over a shell of momenta, as above; since the couplings Z_g and Z_ϕ are modified, this is followed by
(2) a rescaling as in (9) with parameter $b = 1 - \frac{1}{2}\eta_g \delta t$, and
(3) a rescaling as in (9) with parameter $c = 1 - \frac{1}{2}\eta_\phi \delta t$, restoring the unit value Z_ϕ.

3.3. Higgs units

A third possibility is to use the Higgs mass as a unit of mass; we will call these 'Higgs units'. Note that while the Higgs mass is not known, the mass of the electron, which differs from the Higgs mass by the ratio of the coupling λ to a Yukawa coupling, is at the basis of atomic spectroscopy and hence of modern metrology. Thus, the Higgs units defined here are conceptually closely related to ordinary metric units. By means of a transformation as in (9), with parameters $b = \sqrt{Z_\phi}/m_H$ and $c = 1/\sqrt{Z_\phi}$ we can define the action in Higgs units

$$\begin{aligned}\Gamma''_{k''}(g''_{\mu\nu}, \phi''; \Lambda'', Z''_g, \lambda'', \xi'', \dots) &= \Gamma_{k''}(g''_{\mu\nu}, \phi''; \Lambda'', Z''_g, 1, 1, \lambda'', \xi'', \dots)\\ &= \Gamma_k\big(\phi, g_{\mu\nu}, \Lambda, Z_g, Z_\phi, m^2_H, \lambda, \xi, \dots\big),\end{aligned} \tag{14}$$

with

$$\begin{array}{llll} k'' = \frac{\sqrt{Z_\phi}}{m_H}k, & g''_{\mu\nu} = \frac{m^2_H}{Z_\phi} g_{\mu\nu}, & \phi'' = \frac{Z_\phi}{m_H}\phi, & \Lambda'' = \frac{Z^2_\phi}{m^4_H}\Lambda, \\ Z''_g = \frac{Z_\phi}{m^2_H} Z_g, & \lambda'' = \frac{1}{Z^2_\phi}\lambda, & \xi'' = \frac{1}{Z_\phi}\xi, \dots. & \end{array} \tag{15}$$

Again, the action depends explicitly on k'', which is the cutoff measured in Higgs units.

An infinitesimal RG transformation of Γ'' consists of the following:

(1) functional integration over an infinitesimal shell of momenta, as above. Since this modifies both m_H^2 and Z_ϕ, it is followed by
(2) a rescaling as in (9) with parameter $b = 1 - \frac{1}{2}\eta_{m_H^2}\delta t + \frac{1}{2}\eta_\phi \delta t$, where $\eta_{m_H^2}$ is defined as in (5), and
(3) a rescaling as in (9) with parameter $c = 1 - \frac{1}{2}\eta_\phi \delta t$, restoring the unit value Z_ϕ.

This list does not exhaust all possibilities. For example, one could use the cosmological constant as a unit of mass. This is not a practical choice, because the cosmological constant is not known to any good approximation, but it is a rather natural choice in principle (see equations (A.5)–(A.9) in the appendix) and it is used in lattice calculations (see e.g. [12]). I leave it to the reader to work out the transformations of the fields and couplings in this case.

It is important to underline that the use of these complete RG transformations on $\bar{\mathcal{Q}}$ is entirely equivalent to working with the 'basic' RG transformations on $\mathcal{Q}$. For example the complete RG transformation of the scalar mass parameter m_H^2 in cutoff units is

$$\tilde{\delta} m_H^2 = \beta_{m_H^2}\delta t - 2m_H^2 \delta t - \eta_\phi m_H^2 \delta t. \tag{16}$$

The extra terms which come from steps 2 and 3 cancel out when we compute the transformations of the dimensionless couplings listed in equation (11). For example, in the case of the mass, using the definition of $\tilde{m}_H^2$ in (11), since $\tilde{\delta} k = 0$ and $\tilde{\delta} Z_\phi = 0$ by definition, we find

$$\delta \tilde{m}_H^2 = \frac{\tilde{\delta} m_H^2}{Z_\phi k^2} = \tilde{\delta}\tilde{m}_H^2, \tag{17}$$

which means that the complete RG transformation of $\tilde{m}_H^2$ is given just by the 'basic' RG transformation, provided it is applied also to the factors k^2 and Z_ϕ appearing in the denominator.

Similar remarks apply to the case of the other systems of units. For example in Planck units the complete transformation of the mass is

$$\delta' m_H^2 = \beta_{m_H^2}\delta t - \eta_g m_H^2 \delta t - \eta_\phi m_H^2 \delta t. \tag{18}$$

In Planck units, the total transformations $\delta' Z_g = 0$ and $\delta' Z_\phi = 0$ by design, so the complete transformation of the dimensionless 'mass in Planck units' $m_H'^2$ defined in (13) is

$$\delta m_H'^2 = \frac{\delta' m_H^2}{16\pi Z_g Z_\phi} = \delta' m_H'^2, \tag{19}$$

meaning that the complete RG transformation of $m_H'^2$ can be computed just from the 'basic' RG transformation, provided it is applied also to the factors Z_g and Z_ϕ in the denominator. Similar considerations apply also to the other primed and double primed couplings.

The gravitational RG has been studied intensively in the last 10 years with the main aim of proving nonperturbative renormalizability of the theory along the lines of [13]. The derivations of the beta functions for gravity in the Einstein–Hilbert truncation [14] and gravity coupled to matter in [15], as well as their applications in [16, 17] to the search of a gravitational FP, were based on the exact RG equation, taking into account only the first of the three steps described above for each choice of unit. The same applies to subsequent calculations that take into account higher powers of the curvature [18]. By the arguments given above, the beta functions found in these papers for the dimensionless ratios $\tilde{G} = Gk^2$ and $\tilde{\Lambda} = \Lambda/k^4$ are the complete beta functions in cutoff units and do not need any further correction term (in particular, the FP is unaffected).

Let us conclude this section by observing that the primed (Planck) and double primed (Higgs) units could be ill-suited for the discussion of FPs. The great virtue of cutoff units is that the vectorfield defined by the beta functions is t-independent. This is not the case with the other choices of units, since, as noted above, in these units the beta functions have an explicit dependence on the cutoff variables k' and k'' respectively. This makes the physical interpretation of the system of equations less transparent [11].

4. The beta functions

Having discussed the general form of the RG equations of the toy model in various systems of units, I will now illustrate the actual behaviour of the couplings, under the assumption that the action (8) gives a good description of physics at the energy scale that is of interest. Beyond this fundamental assumption, any calculation will require a number of approximations. First, we are going to neglect the anomalous dimension η_ϕ. This is supported by the calculations of [19], where it was shown that the anomalous dimension is suppressed by some power of k/m_P below the Planck scale. Furthermore, in the UV limit we see from equation (6.1a) of [19], together with the results of [16, 17], that η_g tends to a constant of order 10^{-3} if there is an FP. From here on we shall simply assume $Z_\phi = 1$ and $\eta_\phi = 0$. In accordance with observations, we will also assume that the cosmological constant is much smaller than all other scales that come into play. Since our main focus is not on explaining the smallness of the cosmological constant, we shall simply assume that this condition is met.

Qualitatively, the overall picture is characterized by the presence of two thresholds which correspond to the physical Higgs mass and the Planck mass. There are therefore three distinct regimes: the low-energy regime $\sqrt{\Lambda} \ll k^2 \ll m_H^2 \ll Z_g$, the intermediate regime $\sqrt{\Lambda} \ll m_H^2 \ll k^2 \ll Z_g$, and the asymptotic UV regime where, in principle, $k^2 \gg Z_g$. The beta functions of the couplings in (8) have been calculated in [17]. They consist of integrals of certain rational functions of the momenta, the cutoff and the couplings. The momentum integrals are peaked at momenta of order k and in each regime, one can pick in the beta functions the dominant terms and discard all the others. This leads to dramatic simplifications.

Let us discuss each regime in turn, in order of increasing energy. In the low-energy regime we have $\beta_{Z_g} = a_1 k^2$, with $a_1 = -\frac{29}{384\pi^2}$, when using a suitable cutoff function. The solution of the flow equation has the form

$$Z_g = \bar{Z}_g \left(1 + \frac{1}{2} a_1 \frac{k^2}{\bar{Z}_g}\right), \tag{20}$$

where $\bar{Z}_g = Z_g(k = 0)$. By definition in this regime the initial value $\bar{Z}_g$ is much larger than k^2 so the second term in (20) is very small with respect to the first and we can approximate Z_g by the constant $\bar{Z}_g$. The anomalous dimension η_g is consequently also very small. When we look at the beta functions of the other couplings, they are also suppressed by powers of k^2/Z_g, so we can effectively approximate m_H^2 by a constant $\bar{m}_H^2$ and λ by a constant $\bar{\lambda}$. Altogether, in the low-energy regime there is no significant running of the couplings.

Eventually, as k^2 grows, it overcomes the threshold at $k^2 \approx \bar{m}_H^2$. In the intermediate regime where $\bar{m}_H^2 \ll k^2 \ll Z_g$, the beta function of Z_g is the same as in the low-energy regime, except for the numerical coefficient $a_1 = \frac{-33+24\xi}{384\pi^2}$. The difference is due to the contribution of the scalar field, which is felt only above the mass threshold. Since k^2 is still much smaller than Z_g, we can still assume that Z_g is almost constant and equal to $\bar{Z}_g$. In the intermediate regime the dominant term in the beta function of the mass is $\beta_{m_H^2} = c_1 \lambda k^2$, with $c_1 = \frac{3}{4\pi^2}$. This is essentially the one-loop result for a quartically self-interacting scalar theory; the gravitons do not contribute significantly to this beta function in this regime. If we assume that the threshold occurs exactly at $k^2 = \bar{m}_H^2$, matching the solution in the intermediate regime to that in the low-energy regime we have

$$m_H^2(k) = \bar{m}_H^2 \left(1 + \frac{1}{2}\lambda c_1 \frac{k^2 - \bar{m}_H^2}{\bar{m}_H^2}\right) \approx \frac{1}{2} c_1 \lambda k^2. \tag{21}$$

This running is a reflection of the quadratic divergences that occur in perturbative calculations. The dominant term in the beta function of λ is $\beta_\lambda = d_1 \lambda^2$ with $d_1 = \frac{9}{8\pi^2}$. The contribution of gravitons is negligible. The solution is

$$\lambda(k) = \frac{\bar{\lambda}}{1 - d_1 \bar{\lambda} \ln \frac{k}{\bar{m}_H}}, \tag{22}$$

where $\bar{\lambda} = \lambda(k = \bar{m})$ is the value of λ at (and below) the threshold. This is the standard one-loop result, exhibiting a Landau pole at $k = \bar{m}_H \, \mathrm{e}^{1/(d_1 \bar{\lambda})}$. The pole can be avoided if $d_1 \bar{\lambda} < 1/\ln(Z_g/\bar{m}_H)$.

Beyond the Planck scale, field theory will make sense if the couplings approach a FP [13]. In this case, it never really happens that $k^2 \gg Z_g$; instead, Z_g scales exactly like k^2, so that the ratio $\tilde{Z}_g = Z_g(k)/k^2$ tends to a constant. In [17] it was found that the model (8) has a 'Gaussian-matter FP' with nonzero values $\tilde{\Lambda}_*$ and $\tilde{G}_*$ while all other couplings are zero. It is useful to emphasize at this point, however, that the subsequent discussion does not depend on the existence of such an FP. The RG behaviour we have encountered in the intermediate regime would still remain valid even if the field theoretic description was only an effective one, with a physical cutoff somewhere near the Planck scale.

Let us now summarize the consequences of these findings for the parameter α. In the low-energy regime there is no significant running of any quantity, so Higgs units and Planck units simply differ by a fixed multiplicative factor:

$$\bar{\alpha} = \frac{\bar{m}_H}{\sqrt{16\pi \bar{Z}_g}} = \bar{m}_H \sqrt{\bar{G}} \approx 10^{-16}. \tag{23}$$

This is the regime where all of current particle physics takes place. In the intermediate regime, Z_g is still essentially constant but m_H^2 runs, to leading order like k^2. From equation (21), keeping only the leading terms, we find

$$\alpha \approx \sqrt{\frac{c_1 \lambda \bar{G}}{2}} k. \tag{24}$$

In the asymptotic UV regime, assuming that the theory approaches a UV FP, α would tend again to a constant

$$\alpha_* = \frac{\tilde{m}_{H*}}{\sqrt{16\pi \tilde{Z}_{g*}}} = \tilde{m}_{H*} \sqrt{\tilde{G}_*}, \tag{25}$$

where the asterisk denotes the values of the variables at the FP. The value of α_* clearly depends on the details of the FP. In [17] we searched for an FP with nonzero values of the scalar couplings, but we were unable to find one. It seems likely that if a potential exists with nonzero mass, it is not polynomial. However, this issue remains unsettled. In the case of the 'Gaussian-matter FP', after reaching a maximum around the Planck scale, the ratio α tends to zero for $t \to \infty$.

To conclude this section, some cautionary comments are in order. It is not granted that the action (8) is a good description of the world at any energy scale, much less that is a good description at *all* energy scales. In fact, experience with other QFTs leads us to expect that if (8) has anything to do with the real world in the UV regime, then it is unlikely to do so at lower energies. In this respect, QCD is probably a useful guide: it has a very simple description in terms of quark and gluon fields at high energy, near its UV FP, but this description becomes practically useless at energies of the order of 1 GeV, where the strong coupling becomes strong and other terms in the action become important. Likewise, one would expect that if a gravitational FP is governed by an action like (8), perhaps containing a few other terms, then at and below the Planck scale the action, in terms of the variables $g_{\mu\nu}$ and ϕ, will be an extremely complicated, possibly nonlocal functional.

This is in contrast to the common expectation that a functional like (8) may be a reasonable approximation in the low-energy and possibly in the intermediate regime (this expectation comes from the fact that the gravitational interaction is proportional to the external momenta in Feynman diagrams, so that perturbation theory works well with the action (8) at sub-Planckian energies). It is possible that this expectation is correct and that there is no UV FP. Then, the preceding analysis could still be applied in the low-energy and intermediate regime. Alternatively, a possible reconciliation of these points of view may come through a field redefinition: at intermediate/low energies the action may again have a general structure similar to (8), but in terms of other variables $\gamma_{\mu\nu}$ and φ, related to $g_{\mu\nu}$ and ϕ by some functional field redefinition (note that one always expects to have some metric in the action). Examples of such transformations are already well-known in scalar–tensor theories [20]. This would parallel the transition from quark/gluon to meson/baryon degrees of freedom in QCD at low energy [9].

5. AdS space, from the bottom up

In the preceding section, we have seen the 'basic' beta functions on $\mathcal{Q}$ for the toy model. Let us now see what the complete RG flow on $\bar{\mathcal{Q}}$ looks like for various choices of units. This will allow us to make contact with the RS model.

For simplicity we concentrate on the second, third and fourth terms of the action (8), which carry all the relevant information. To simplify the discussion, we also assume that the three regimes described in section 4 extend all the way to the thresholds. We choose $t = 0$ at the Planck scale, so that $t = \log(k/m_P)$. Let us assume that at the Planck scale the scalar mass has some 'natural' value of the order of the Planck mass: $m_H^2(t = 0) \approx Z_g$. In the rest of this section we will consider only the leading scaling behaviour of each quantity in the intermediate regime, neglecting the anomalous dimensions η_g and η_ϕ, and the logarithmic running of λ.

In cutoff units at the Planck scale the action reads

$$\tilde{\Gamma}|_{t=0} = \int \mathrm{d}^4x\sqrt{\tilde{g}}\left[\tilde{Z}_g R[\tilde{g}] - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \frac{1}{2}\tilde{\lambda}(\tilde{\phi}^2 - \tilde{\upsilon}^2)^2\right]. \tag{26}$$

All couplings are evaluated at $t = 0$. Let us now run the RG in cutoff units, as described in section 3, towards lower energies (negative t). As discussed in section 4, Z_g does not run significantly; for the mass we keep only the term proportional to k^2 and neglect any constant. Then we have

$$\tilde{Z}_g(t) = \frac{\bar{Z}_g}{k^2} = \tilde{Z}_g(0)\,\mathrm{e}^{-2t}, \qquad \tilde{m}_H(t)^2 = \tilde{m}_H(0)^2. \tag{27}$$

Since we neglect the running of λ, also $\tilde{\upsilon}(t)^2 = \tilde{\upsilon}(0)^2$. Furthermore, since $\eta_\phi \approx 0$, the third step of the RG transformation does not change the fields. However, the second step of the RG transformations is nontrivial: it generates a rescaling of the metric and of the scalar field with a suitable power of the factor $1 - \delta t$ at each RG step:

$$\frac{\mathrm{d}\tilde{g}_{\mu\nu}}{\mathrm{d}t} = 2\tilde{g}_{\mu\nu}, \qquad \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t} = -\tilde{\phi}, \tag{28}$$

which yields

$$\tilde{g}_{\mu\nu}(t) = \mathrm{e}^{2t}\tilde{g}_{\mu\nu}(0), \qquad \tilde{\phi}(t) = \mathrm{e}^{-t}\tilde{\phi}(0). \tag{29}$$

This rescaling has to be applied to all the unintegrated Fourier modes of the field, having momenta lower than k. In practice we are mostly interested in the zero-momentum modes (the constant scalar and a flat metric) and (29) implies that these modes have to be rescaled with t as shown.

Taking into account equations (27) and (29), the action at some lower scale $t < 0$ is given by

$$\begin{aligned}\tilde{\Gamma}_t &= \int \mathrm{d}^4x\sqrt{\tilde{g}(t)}\left[\tilde{Z}_g(t)R[\tilde{g}(t)] - \frac{1}{2}\tilde{g}^{\mu\nu}(t)\partial_\mu\tilde{\phi}(t)\partial_\nu\tilde{\phi}(t) - \frac{1}{2}\tilde{\lambda}(t)(\tilde{\phi}(t)^2 - \tilde{\upsilon}(t)^2)^2\right]\\ &= \int \mathrm{d}^4x\sqrt{\tilde{g}(0)}\left[\tilde{Z}_g(0)R[\tilde{g}(0)] - \frac{1}{2}\tilde{g}^{\mu\nu}(0)\partial_\mu\tilde{\phi}(0)\partial_\nu\tilde{\phi}(0) - \frac{1}{2}\tilde{\lambda}(0)(\tilde{\phi}(0)^2 - e^{2t}\tilde{\upsilon}(0)^2)^2\right].\end{aligned} \tag{30}$$

In Planck units, the details of the calculation differ but the final result is the same. The action at the Planck scale is given by

$$\Gamma'|_{t=0} = \int \mathrm{d}^4x\sqrt{g'}\left[\frac{1}{16\pi}R[g'] - \frac{1}{2}g'^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' - \frac{1}{2}\lambda'(\phi'^2 - \upsilon'^2)^2\right]. \tag{31}$$

In these units, by definition $Z'_g = \frac{1}{16\pi}$ independent of t. Since we neglect η_g, the second step of the RG is trivial. Since we neglect η_ϕ, the third step of the RG is trivial. The mass runs like

$$m'_H(t)^2 = m'_H(0)^2\,\mathrm{e}^{2t}. \tag{32}$$

Everything else, including field normalizations, is t-independent within the approximations that we made, so in these units the scale-dependence of the mass is immediately evident. The reader can check that, due to linear running of the mass, Higgs units work like cutoff units.

We can then compare these results to the original RS model. One sees that the calculations leading from equations (26) to (30) are identical to the calculations leading from equations (17) to (19) in [5]. This should perhaps not come as a surprise, in view of the 'holographic' interpretation of the RS model [7]. Their five-dimensional metric contains a one-parameter family of four-dimensional metrics with an exponentially varying warp factor:

$$g_{\mu\nu}(x,t) = \mathrm{e}^{at}\eta_{\mu\nu}. \tag{33}$$

We see that essentially the same one-parameter family of metrics is generated in a purely four-dimensional theory between the Higgs and the Planck scales, when one works in cutoff units (or in Higgs units).

To make the connection more complete, we should say how distances have to be defined in the space R^+ parametrized by the scale k. Since k is typically identified with some momentum variable appearing in a process, it may seem natural to choose the 'proper distance' in cutoff space to be proportional to $\mathrm{d}k$, as would befit a linear variable. But the space R^+ is not a linear space and this is not the natural choice. To motivate this from a physical point of view, observe that the units that we have chosen are subject to RG flow, so the question arises: when we measure a mass m in a certain system of units, at what scale should the unit be taken? The natural, or 'intrinsic', definition is to take $k = m$. This is the only way to define a real number out of a mass and a running unit, without making reference to another mass scale. Thus, when we 'measure' a physical mass m in Higgs units the real number that is obtained is the ratio

$$\frac{m}{m_H(m)}. \tag{34}$$

The physical rationale of this assumption is that when we perform a measurement of a mass parameter, we are not really measuring just that mass, but also the unit of mass. The two measurements cannot be disentangled and all one obtains is a value for the ratio between the two; if m is the only scale in the system, (34) follows. Similarly a measurement of a length ℓ gives the number $\ell m_H(\ell^{-1})$. On the other hand a measurement of a mass m in Planck units gives the number

$$m\sqrt{G(m)}, \tag{35}$$

where Newton's constant is evaluated at the scale $k = m$, while a measurement of a length ℓ yields $\ell/\sqrt{G(\ell^{-1})}$, where G is evaluated at the scale $k = \ell^{-1}$.

This natural assumption leads to unfamiliar consequences. Let ℓ_1 and ℓ_2 be two lengths and $r = \ell_2/\ell_1$ their dimensionless ratio. The ratio of the two real numbers which give the values of the lengths in the two systems of units will not in general be equal to each other, nor to r. According to our postulate, the ratio of the two lengths in Planck units is

$$r_P = \frac{\ell_2}{\sqrt{G\left(\ell_2^{-1}\right)}}\frac{\sqrt{G\left(\ell_1^{-1}\right)}}{\ell_1} = \sqrt{\frac{G\left(\ell_1^{-1}\right)}{G\left(\ell_2^{-1}\right)}}r. \tag{36}$$

Similarly, the ratio of the two lengths in Higgs units, is

$$r_H = \frac{\ell_2 m_H\left(\ell_2^{-1}\right)}{\ell_1 m_H\left(\ell_1^{-1}\right)} = \frac{m_H\left(\ell_2^{-1}\right)}{m_H\left(\ell_1^{-1}\right)}r = \frac{\alpha\left(\ell_2^{-1}\right)}{\alpha\left(\ell_1^{-1}\right)}r_P. \tag{37}$$

So, if the conversion factor α is not the same at all scales, a set of masses that are linearly spaced when referred to Planck units will not be linearly spaced when referred to Higgs units. An extreme example of this fact has already been noticed in [11]: if the QFT of gravity has a fixed point, when $t \to \infty$ the variable k' defined in (13), which gives the value of the cutoff in Planck units, must tend to a finite limit $k'_* = \sqrt{\tilde{G}_*}$, whereas the variable k'' defined in (15), which gives the value of the cutoff in Higgs units, may well tend to infinity.

All this is unusual, but it does not imply any inconsistency. It only means that cutoff space cannot be thought of as a linear space, but should be treated as a one-dimensional manifold[4]. The dimensionless variables t, k' and k'' provide coordinate systems on this manifold, related to the use of cutoff, Planck and Higgs units respectively. The relation between these variables is nonlinear and may not even be invertible. Since by construction t grows as the RG transformations are iterated, the direction of growing t defines a natural orientation in cutoff space, but the directions of growing k' and k'' may or may not agree with this orientation[5].

Since cutoff space does not have a natural linear structure, a proper notion of distance between, say, the Higgs scale and the Planck scale requires that a metric be given. The most

[4] There may be a connection with so-called doubly special relativity models, where the Lorentz group is postulated to act in a nonlinear way on momentum space [21].

[5] For example, it has been observed in [11] that in the approach to the UV FP, Newton's constant follows damped oscillations. At each oscillation, there would be two stationary points where k' could not be used as a coordinate, and the direction of growing k' disagrees with the direction of growing t at each second half-cycle. An alternative coordinate better adapted to the FP behaviour has been proposed in [11]. Similarly in Higgs units, assuming that $\tilde{m}_H \to 0$ at the FP, there would be a scale of the order of Planck's scale where $\tilde{\beta}_{\tilde{m}_H}$ changes sign. Below this scale k'' would be an increasing function of t and above this scale it would be decreasing.

natural choice of distance in R^+ would be given by counting the number of iterations of the RG transformation. For continuous RG transformations ($\delta t \to 0$) the proper distance is then simply proportional to $\mathrm{d}t$. Note that this logarithmic scale is also more natural insofar as it places the infrared limit (infinitely large systems) at infinite distance from any finite system.

At this point, we have all the ingredients to reconstruct the entire five-dimensional RS metric from the four-dimensional RG. Combining the metric on spacetime with the metric on cutoff space, together with the assumption that cutoff and physical space are orthogonal, produces a metric on a five-dimensional manifold. In the slice between the Higgs and the Planck scales, considering only the leading behaviour (21), this metric is locally the AdS metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{e}^{2t}\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{38}$$

In this way, one can entirely reconstruct the RS five-dimensional space from four-dimensional data, without having to postulate that the fifth dimension is physically accessible.

6. Conclusions

In this paper, I have discussed the way in which field rescalings affect the RG flow in the presence of gravity. One general lesson that can be drawn from this discussion is that in certain circumstances the use of dimensionful quantities could be misleading. The measurement of a dimensionful quantity is in reality the measurement of the ratio between that quantity and some unit. In daily life we can refer to a well-established system of units, which have been chosen for their availability and stability under a wide range of circumstances; in particular, they are unaffected by RG ambiguities. However, their use in extreme situations and in particular in quantum gravity may require some care. The relation between a certain dimensionful quantity measured in a high-energy experiment and standard metric units can be very indirect [4]. In order to avoid potential ambiguities, the safest procedure is to consider only dimensionless ratios, with the understanding that any physical measurement is a simultaneous measurement of the numerator and the denominator, and that one may not be able to measure the numerator and denominator separately. In theoretical calculations, one has to take into account the fact that both numerator and denominator may be subject to RG flow.

A related point is the following. As long as we study a theory describing a limited set of physical phenomena, we can always ignore these issues by taking as a unit something external to the theory. For example, if we study only strong interactions we can take the mass of the Z as an absolute, external unit; if we study only electroweak and strong interactions, the Planck mass defines an absolute, external unit. If on the other hand we want to include in the theory also gravitational phenomena, then also the Planck mass will be subject to RG flow, as we have seen, and there is no external parameter that can be taken as an absolute unit. In these circumstances, the safest procedure seems to be to avoid statements about dimensionful quantities.

To describe physics entirely in terms of dimensionless quantities would be similar to describing a gauge theory entirely in terms of gauge invariant quantities, and could become very cumbersome. However, when dimensionful variables are used, one should take care to avoid ambiguous statements. For example, the statement $k^2 \ll 1/G$ is a statement about dimensionful quantities that is unambiguous, because it is equivalent to the statement $k^2 G \ll 1$. On the other hand, a statement such as '$k \to \infty$' (which is usually taken to define the ultraviolet limit) is ambiguous. To see why, consider for example the asymptotic UV regime of the toy model, which has been briefly mentioned in section 4. There is evidence for the existence of a 'Gaussian-matter FP' where Newton's constant, in cutoff units, tends to a nonzero value $\tilde{G}_*$

and the ratio m_H/k tends to zero [17]. This would imply that in the UV limit the cutoff in Higgs units $k'' \to \infty$, while the cutoff in Planck units k' tends to a finite limit k'_*. In such a case saying that '$k \to \infty$' is at best misleading. An unambiguous way of referring to the UV limit could be to say '$t \to \infty$', meaning that the RG transformation is iterated infinitely many times. Similarly, to say that a dimensionful coupling such as Newton's constant is asymptotically free is in itself meaningless. One has to specify in what units Newton's constant is being measured. Depending on the flow and on the choice of units, it could tend to zero, to a finite constant or to infinity (for example, Newton's constant is always equal to one in Planck units, so the statement above can evidently never be true in Planck units).

Let us now discuss possible connections with the hierarchy problem. As mentioned in the introduction, the problem posed by the smallness of the ratio (1) in the toy model is conceptually very similar to that posed by the smallness of all lepton and quark masses, in Planck units. The mass of composite particles poses a different problem. For example, the mass of the proton is $m_p \approx 10^{-19} m_P$, but the origin of this small ratio must be to a large extent independent of the hierarchy problem represented by (1). This is because the mass of the proton originates mostly from the kinetic energy of the quarks and from the energy of the gluon field that binds them, and only in minimal part from the rest mass of its pointlike constituents. We are now close to having a satisfactory explanation for this problem, based on RG effects in QCD [22]. The mass of the proton corresponds to the scale at which the strong coupling α_s becomes large enough to bind the quarks. This requires

$$\alpha_s(m_p) \approx 1. \tag{39}$$

Starting from a relatively small value for α_s at the unification scale m_U (which is close to the Planck scale), due to the logarithmic running, one has to go to extremely small scales before the coupling becomes strong. To make this a bit more quantitative, solve the RG equation

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}t} = -b\alpha_s^2, \tag{40}$$

where $b = \frac{1}{2\pi}\left(11 - \frac{2}{3}n_f\right)$ is the beta function in the presence of n_f flavours of quarks, and use the condition (39). One gets

$$m_p = m_U\, \mathrm{e}^{-\frac{1}{b}\left(\frac{1}{\alpha_s(m_U)}-1\right)}. \tag{41}$$

Without entering into details of specific grand unification schemes, it clear that very small ratios can be generated in this way.

As pointed out before, the masses of hadrons originate from strong interaction physics and those of pointlike particles from electroweak physics. The success of the RG of QCD in producing a small ratio for m_p/m_P does not imply that a similar mechanism can be found for the masses of the Higgs scalar or the electron, but the idea remains extremely attractive. Insofar as the RS model provides a mechanism for generating small mass ratios, the ability of the four-dimensional RG to reproduce its main results seems to be encouraging. However, to reproduce the RS model from the RG we assumed that the running mass is proportional to k; more precisely, we have neglected any constant term, which is permitted in the solution of the RG equation. When one solves the flow equations starting at m_U, there is no reason to assume that the constant term is negligible, so reproducing the trajectory (21) requires an *a priori* fine tuning of the initial conditions at the Planck or GUT scale. In the presence of fermion fields there may exist a dynamical mechanism, based essentially on an approximate IR fixed point for the scalar mass, that attracts it linearly towards the origin [23]. This mechanism demands a large number of scalar and fermion fields and/or large values of the Yukawa couplings. A slow decrease of the Yukawa couplings would then switch off this running at some small scale, similar to the effect described above for QCD. It thus seems worth looking in more detail at the RG flow of the model, perhaps taking into account other parameters or other matter fields, and searching for RG trajectories that exhibit an extended intermediate regime. It would be natural to explore these issues under the assumption of the existence of a UV FP, though this hypothesis may not be necessary for an RG mechanism to work.

Another extension of this work would be to consider the effect of more general classes of field redefinitions, for example those involving conformal rescalings of the metric. Presumably, one could relate this to extensions of the RS model where the branes are not parallel [24]. We note that in this generalization the scale transformations of dimensional analysis would have the status of gauge transformations.

It is also worth comparing these RG arguments to Dirac's original prediction that G, when measured in atomic units, would decrease like the inverse of the epoch τ. One can reasonably assume that in a cosmological context the cutoff k can be identified with the inverse of the epoch [25] (or maybe some function thereof). Now the value of G in Higgs units is $Gm_H^2 = \alpha^2$. Therefore, in the intermediate regime it should behave like τ^{-2}, and furthermore it should stop running after an epoch of the order m_H^{-1}. So, if this picture is correct, Newton's constant measured in Higgs units would indeed be time dependent, but only in the very early universe, before the electroweak phase transition.

We have seen that within certain approximations the AdS geometry can be interpreted as a description of the quadratic divergences occurring in the toy model. For a review of holography and its relation to other important current developments in theoretical physics see [26]. I would like to emphasize once again that this reconstruction of the AdS metric is purely formal, whereas in the RS model it is assumed that it is in principle possible (at least for gravitons) to propagate into the fifth dimension. Furthermore, while RS derive the RG flow from the five-dimensional Einstein equations, we used a standard four-dimensional beta function calculation. The phenomenology of the RS model, as well as that of other higher-dimensional theories that place the Planck scale close to the Higgs scale, is very different from that of the purely four-dimensional model discussed here. Since t is not interpreted as a physical dimension, there are no Kaluza–Klein modes and no radion field. Insofar as these higher-dimensional theories may be amenable to experimental verification, the four-dimensional RG approach discussed here provides an experimentally distinguishable alternative.

In conclusion, let us observe that until now particle physics has been explored in the low-energy regime, where units can be assumed not to run significantly. Particle accelerators are now on the verge of entering into the intermediate regime, where the metric in Higgs units ($g''_{\mu\nu}$) will begin to scale strongly and will differ from the metric in Planck units ($g'_{\mu\nu}$) by a scale-dependent factor. While this is not what one would ordinarily call a quantum gravity effect, it is a quantum effect, and it affects our understanding of the spacetime structure. It will be interesting to see if this can have any observable consequences.

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Appendix. Examples of essential and inessential couplings

If a Lagrangian is parametrized in an arbitrary way, in general no parameters can be straightforwardly absorbed into a field redefinition. This is the case, for example, in a scalar field theory with the following action:

$$S = \int d^4x \left[-\frac{Z}{2}\partial_\mu\phi\partial^\mu\phi + \lambda_2\phi^2 + \lambda_4\phi^4 + \lambda_6\phi^6 \ldots \right]. \tag{A.1}$$

None of the parameters appearing in this action satisfies the criterion for inessentiality given in [13], namely that the variation of the Lagrangian with respect to the parameter vanishes or is a total derivative on shell. In fact, all parameters scale nontrivially under the c-transformations in (9).

If we redefine $\lambda_{2n} = Z^n\rho_{2n}$ the action becomes

$$S = \int d^4x \left[-\frac{Z}{2}\partial_\mu\phi\partial^\mu\phi + Z\rho_2\phi^2 + Z^2\rho_4\phi^4 + Z^3\rho_6\phi^6 \ldots \right]. \tag{A.2}$$

In this parametrization the couplings ρ_{2n} are invariant under the c-transformations, while Z is not. This is therefore an adapted parametrization of the action. The wavefunction renormalization is inessential and indeed it can be absorbed by a simple rescaling $\sqrt{Z}\phi = \psi$:

$$S = \int d^4x \left[-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi + \rho_2\psi^2 + \rho_4\psi^4 + \rho_6\psi^6 \ldots \right]. \tag{A.3}$$

It is important to stress that this statement refers to the parametrization (A.2): one cannot say that Z in the action (A.1) is inessential, because by this token one could conclude that *every* parameter appearing in (A.1) is inessential. For example, one could define alternatively $\lambda_{2n} = \lambda_2^n\sigma_{2n}$ and $Z = \lambda_2\zeta$, in which case the action becomes

$$S = \int d^4x \left[-\frac{\zeta}{2}\lambda_2\partial_\mu\phi\partial^\mu\phi + \lambda_2\phi^2 + \lambda_2^2\sigma_4\phi^4 + \lambda_2^3\sigma_6\phi^6 \ldots \right], \tag{A.4}$$

and in this parametrization it is the mass λ_2 that is inessential and disappears by the rescaling $\sqrt{\lambda_2}\phi = \Phi$.

Similar considerations apply also to gravity, with the additional twist that in this case eliminating an inessential parameter is equivalent to a choice of units of mass. If we write a general action, somewhat symbolically, as

$$S = \int d^4x\sqrt{g}[a_0 + a_1R + a_2R^2 + a_3R^3 + \ldots] \tag{A.5}$$

we can use the scale invariance with parameter b in equation (9) to eliminate one of the couplings. For example we can redefine $a_n = a_1^{2-n}a'_n$, $(a'_1 = 1)$ leading to

$$S = \int d^4x\sqrt{g}\left[a_1^2a'_0 + a_1R + a'_2R^2 + a_1^{-1}a'_3R^3 + \ldots\right] \tag{A.6}$$

then rescaling $g_{\mu\nu} = a_1^{-1}g'_{\mu\nu}$, a_1 disappears

$$S = \int d^4x\sqrt{g'}[a'_0 + R' + a'_2R'^2 + a'_3R'^3 + \ldots], \tag{A.7}$$

which (up to trivial factors) is the action in Planck units. So a_1 is inessential in the parametrization (A.6). Alternatively we can redefine $a_n = a_0^{\frac{(2-n)}{2}}a''_n$, $(a''_0 = 1)$, so that

$$S = \int d^4x\sqrt{g}\left[a_0 + \sqrt{a_0}a''_1R + a''_2R^2 + \frac{a''_3}{\sqrt{a_0}}R^3 + \ldots\right]. \tag{A.8}$$

Then defining $g_{\mu\nu} = \frac{1}{\sqrt{a_0}}g''_{\mu\nu}$, a_0 disappears

$$S = \int d^4x\sqrt{g''}[1 + a''_1R'' + a''_2R''^2 + a''_3R''^3 + \ldots] \tag{A.9}$$

This is the action written in units of the cosmological constant. Note that one cannot eliminate a_2, which is invariant under b-transformations, but one could eliminate a_3 or any one of the higher couplings.

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